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LETTER TO THE EDITOR

The Yang–Lee edge singularity by the phenomenological renormalisation group

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Abstract. The phenomenological renormalisation group is applied to the 1D transverse Ising model with longitudinal imaginary field in order to investigate the Yang–Lee edge singularity of the equivalent 2D Ising model. The value found for the exponent ν is $\nu = 0.42 \pm 0.005$, in agreement with previous series expansion estimates. The method is also extended to calculate directly the exponent σ of the magnetisation.

Recently much interest has been given to the phase transition in the presence of a purely imaginary symmetry-breaking field (see, for example, Kurtze 1980). This transition, which appears to be of second order for Ising and general n-vector models, belongs to a different universality class than a zero-field transition. While the latter one corresponds to the cumulation of zeros of the partition function near the real axis in the complex field plane, the former, usually called the Yang–Lee edge singularity, corresponds to the cumulation of those zeros around some particular point on the imaginary axis when the temperature is above the critical temperature (Yang and Lee 1952). The transition is then described through the critical behaviour of the zero-density function which is proportional to the real part of the spontaneous magnetisation:

$$g(h) \sim \operatorname{Re} M(h) \sim (h - h_c)^{\sigma}. \tag{1}$$

h is the imaginary field and h_c is the temperature-dependent critical field below which $g(h)$ is zero. As the critical behaviour is the same whether varying the temperature or the imaginary field (critical exponent $\Delta = 1$), $\beta = 1/\delta = \sigma$ and only one exponent is sufficient to characterise all the critical singularities.

The interest of studying the Yang–Lee edge singularity is two-fold: first, it should influence the equation of state in the limit where the edge of the zero-density gap approaches the real axis; second, it opens a new universality class which could contain some other problems (with real parameters) as it was already shown for branching polymers in $d+2$ dimensions (Parisi and Sourlas).

Except in two trivial cases $d = 1$ and $d = \infty$, where the exact solutions are available and give respectively $\sigma = -\frac{1}{2}$ (Yang and Lee 1952) and $\sigma = \frac{1}{2}$ (Baker and Moussa 1978) the problem of the Yang–Lee edge singularity has only been treated by approximate methods. The Field theoretical renormalisation group was applied in $d = 6 - \varepsilon$ dimensions (Fisher 1978) and series expansion results are available for $d = 2$ and $d = 3$ (Kortman and Griffiths 1971, Kurtze and Fisher 1979). Recently we applied the real space renormalisation group to the analogous quantum problem in order to calculate the Yang–Lee singularity in $d = 2$ (Uzelac *et al* 1979, 1980). Unlike the results for zero field, obtained by the same method, those for the imaginary field show several difficulties in converging to the series expansion results. We observe strong logarithmic oscillations in the real part of the magnetisation near criticality. The value of σ calculated from the non-oscillating imaginary part of the magnetisation, converges towards $\sigma = -0.22 \pm 0.005$ (Uzelac *et al* 1981) which is definitely larger (in absolute values) than the series expansion result $\sigma = -0.163 \pm 0.003$ (Kurtze and Fisher 1979).

In this Letter we report our more recent results using a different renormalisation group (RG) technique, the phenomenological renormalisation group (PRG), whose advantage is to give the critical exponents with very good precision. We calculate the critical exponent ν and obtain a value very close to series expansion results. We have also been able to apply the PRG directly to the order parameter, and the resulting exponent (obtained in such a direct way) can be discussed in connection with previous discrepancies.

The model considered here is the transverse Ising chain in a parallel imaginary field described by the Hamiltonian

$$H = -J \sum_i S_i^x S_{i+1}^x - \Gamma \sum_i S_i^z - \mathrm{i}h \sum_i S_i^x \tag{2}$$

where J, Γ and h are real constants and S^x, S^z are Pauli matrices. The Γ-dependent transition in the ground state of system (2) is equivalent (Suzuki 1976) to the temperature-dependent transition in the two-dimensional classical Ising model in a parallel imaginary field.

The phenomenological renormalisation group formulated by Nightingale (1976) is based on finite size scaling (Fisher and Barber 1972). It has already been applied to the special case of problem (2) when $h/J = 0$ (Sneddon and Stinchcombe 1979, Hamer and Barber 1980). The procedure in this case is as follows. One considers the energy gap ΔE between the ground state and first excited state, which is generally related to the correlation length ξ by $\Delta E \sim 1/\xi^z$. The finite size scaling postulates the same scaling equation for the finite system as for the infinite one. Thus the energy gaps for the blocks of n and m spins are related by

$$\Delta E_n\left(\frac{\Gamma - \Gamma_c}{J}\right) = \left(\frac{m}{n}\right)^z \Delta E_m\left[\left(\frac{n}{m}\right)^{1/\nu}\left(\frac{\Gamma - \Gamma_c}{J}\right)\right] \tag{3}$$

where z and ν are critical exponents of the infinite system.

For problem (2) z is always equal to unity which evolves from the correspondence with the classical 2D problem. Then the critical behaviour can be calculated in a standard way.

The critical field Γ_c is determined from the requirement

$$\Delta E_n(0) = (m/n)\Delta E_m(0) \tag{4}$$

while linearising around Γ_c/J gives the value of the exponent ν.

In this quantum case, the same reasoning applied to the energy gap can easily be extended (Uzelac 1980) to the order parameter in order to calculate directly the corresponding exponents η and β. In the critical point the matrix element of the operator S^x between the first two states $|1\rangle$ and $|2\rangle$ renormalises as

$$\langle 1|S^x|2\rangle_n = (n/m)^{-d_\phi}\langle 1|S^x|2\rangle_m. \tag{5}$$

This gives $\eta = 2d_\varphi + d + z - 2$ and $\beta = d_\varphi \nu$.

The advantage of the PRG is its rapid convergence. While the block methods converge as $1/\lg n$ the present method converges by a power law, which has to be determined for each particular case (cf Derrida 1981). Our analysis of convergence for problem (2) with $h/J = 0$ gives convergence in $1/n^3$ for Γ_c and in $1/n$ for the exponent ν. It is interesting to notice that the PRG applied to the equivalent classical problem also gives the convergence $1/n^3$ for T_c, but $1/n^2$ for ν (Nightingale 1976). The reason for this could lie in the strong 'lattice' anisotropy implicit in the quantum problem and also in our imposing $z = 1$ in equation (3).

In the general case of problem (2), there are two independent parameters, and condition (4) is too weak. It is then reasonable to fix one of the two parameters and apply scaling of type (3) for the other one (differently than in the usual RG procedure). This will induce an additional approximation but the results should still converge to the exact value in the limit $n \to \infty$.

Thus we write the equations analogous to the equations (3) and (4) as a function of h/J fixing the transverse field parameter Γ/J. The exponent is given by

$$\nu = \ln\left(\frac{n}{m}\right)\left[\ln\left(\frac{n}{m}\,\frac{\partial\Delta E_n/\partial(h/J)}{\partial\Delta E_m/\partial(h/J)}\bigg|_{(h_c/J)}\right)\right]^{-1} \tag{6}$$

where h_c/J is the critical field which depends on Γ.

The calculations were performed numerically by taking periodic boundary conditions in the block. $m = n - 1$ has been taken in order to obtain the best convergence. Figure 1 contains the results for the exponent ν as a function of the size n of the block for two different values of the transverse field Γ/J. The convergence is found to be in $1/n^x$, where x is approximately 1.4. Although having slightly different slopes, the two series of points converge to the same value $\nu = 0.42 \pm 0.005$. The resulting value of σ is $\sigma = (d+z)\nu - 1 = -0.164 \pm 0.01$, which is very close to the result $\sigma = -0.163 \pm 0.003$ obtained by series expansion by Fisher and Kurtze (1980).

In order to discuss the discrepancy of our previous result for σ, which was directly derived from the imaginary part of the magnetisation we also performed the direct calculation of σ in the present case. In the case of an imaginary field, equation (5) is replaced by two equations: the real part of the magnetisation is expressed by

$$\langle 1|S^x|2\rangle_n = \left(\frac{n}{m}\right)^{-d_\varphi^{\mathrm{R}}} \langle 1|S^x|2\rangle_m \tag{7}$$

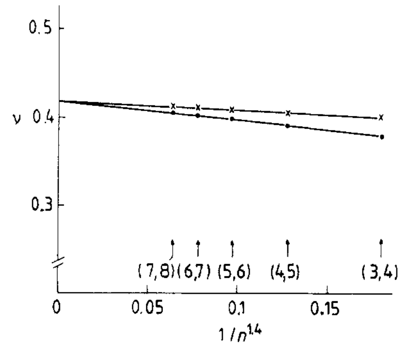

Figure 1. Values of the exponent ν as a function of block size for two different values of Γ_c/J: $\Gamma_c/J = 4$ (●) and $\Gamma_c/J = 10$ (×).

while the imaginary part as the linear response is described by

$$\langle 1|S^x|1\rangle_n = \left(\frac{n}{m}\right)^{-d_\varphi^{\mathrm{I}}} \langle 1|S^x|1\rangle_m. \tag{8}$$

From the scaling argument it follows that $d_\varphi^{\mathrm{I}} = d_\varphi^{\mathrm{R}}$. However, our results give two different values presented in figure 2. The corresponding real part value $\sigma_{\mathrm{R}} = -0.164 \pm 0.015$ coincides with the series expansion result and is coherent with the value obtained from ν, while the imaginary part is close to our previous RG result. The fact that the exponent σ_{I} for the imaginary part is different from that expected should be attributed to some effect which alters the procedure of PRG as well as other RG techniques. As pointed out already (Uzelac *et al* 1979, 1980, 1981), this effect could be due to the influence of the singularity for a finite block, which affects the real and the imaginary part of the magnetisation differently. Then, the value for σ_{I} could be interpreted as intermediate between the 1D value $\sigma = -0.5$ and that of 2D.

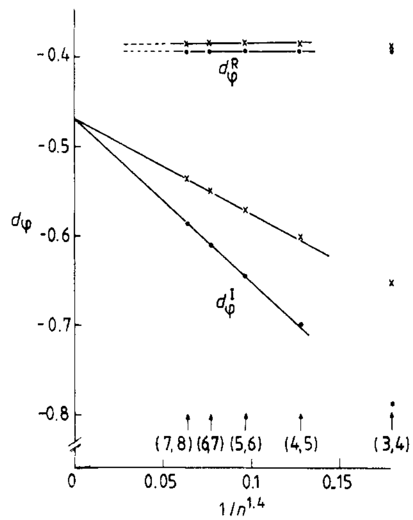

Figure 2. Values for d_φ^{R} and d_φ^{I} as a function of block size for two different values of Γ_c/J: $\Gamma_c/J = 4$ (●) and $\Gamma_c/J = 10$ (×).

In conclusion the PRG was able to give values of the exponents ν and σ for the Yang–Lee edge singularity in 2D in very good agreement with series expansion estimates. However, there still remains the problem of the value of σ when calculated from the imaginary part of the magnetisation which is found to be larger in absolute value.

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